

AN INDECOMPOSABLE AND UNCONDITIONALLY SATURATED BANACH SPACE

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ABSTRACT. We construct an indecomposable reflexive Banach space X_{ius} such that every infinite dimensional closed subspace contains an unconditional basic sequence. We also show that every operator $T \in \mathcal{B}(X_{ius})$ is of the form $\lambda I + S$ with S a strictly singular operator.

1. INTRODUCTION

The aim of this paper is to present a Banach space which is not the sum of two infinite dimensional closed subspaces Y, Z with $Y \cap Z = \{0\}$ and every closed subspace of it contains an unconditional basic sequence. We shall denote this space as X_{ius} . W.T. Gowers' famous dichotomy, [G3], provides an alternative description of this space. Namely X_{ius} is an Indecomposable Banach space not containing any Hereditarily Indecomposable (H.I.) subspace. The problem of the existence of such spaces was posed by H.P. Rosenthal and it is stated in [G2]. The interest for such spaces arises from the coexistence of conditional (indecomposable) and unconditional (unconditionally saturated) structure on them. This is a free translation of W.T.Gowers' comments before stating the problem of the existence of such spaces in [G2] (Problem 5.11). We should mention that Indecomposable spaces which are not H.I. are already known. For example, [AF] provides reflexive H.I. spaces X such that X^* contains an unconditional basic sequence. The methods used in [AF] do not seem to be able to provide H.I. spaces X with X^* unconditionally saturated.

The space presented in this paper is built following ideas used for the construction of H.I. Banach spaces. The method we follow is an adaptation of [AD] constructions as they were extended in [AT]. Both are variations of the fundamental discovery of W.T. Gowers and B. Maurey, [GM]. In our case we use as an unconditional frame a mixed Tsirelson space $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_j]$ which is a space sharing similar properties with Th. Schlumprecht's space S , [S]. The norming set K of the space X_{ius} is a subset of the unit ball of the dual of $T[(\mathcal{A}_{n_j}, \frac{1}{m_j})_j]$. The only difference that the space X_{ius} has from a corresponding construction of a H.I. space concerns the definition of the special functionals. The key observation that changing the special functionals one could obtain interesting non H.I. spaces is due to W.T.Gowers and it was used for the solution of important and long standing problems in the theory of Banach space, [G].

For the space X_{ius} we need the special functionals to be defined such that the following geometric property holds in the space. For every $Y = \langle e_n \rangle_{n \in M}$, $M \in [\mathbb{N}]$, and $(e_n)_{n \in \mathbb{N}}$ the natural basis of X_{ius} , the quotient map $Q : X_{ius} \rightarrow X_{ius}/Y$ is strictly singular. This is equivalent to say that $dist(S_Z, S_Y) = 0$ for all Z infinite dimensional subspace of X_{ius} . This property clearly holds in the case of H.I. spaces. In our case we define the special

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functionals such that the aforementioned property holds and on the other hand we have attempted to keep the dependence inside of each special functional as small as possible. Thus going deeper in the structure of any subspace of X_{ius} the action of the special functionals becomes negligible, which permits us to find unconditional basic sequences. Another property of the space X_{ius} concerns the bounded linear operators. Namely every $T : X_{ius} \rightarrow X_{ius}$ is of the form $T = \lambda I + S$, where S is strictly singular. Thus X_{ius} is not isomorphic to any of its proper subspaces.

2. DEFINITION OF THE SPACE X_{ius}

We shall use the standard notation. Thus c_{00} denotes the linear space of all eventually zero sequences and for $x \in c_{00}$ we denote by $\text{supp}x = \{n : x(n) \neq 0\}$ and by $\text{range}(x)$ the minimal interval of \mathbb{N} containing $\text{supp}x$. Also for $x, y \in c_{00}$ by $x < y$ we mean that $\max \text{supp}x < \min \text{supp}y$. We shall also use the standard results from the theory of bases of Banach spaces as they are described in [LT].

We choose two strictly increasing sequences $(n_j)_j, (m_j)_j$ of positive integers, such that

- (i) $m_1 = 2$ and $m_{j+1} = m_j^5$
- (ii) $n_1 = 4$ and $n_{j+1} = (4n_j)^{s_j}$ where $2^{s_j} \geq m_{j+1}^3$.

Let \mathbf{Q} be the set of scalar sequences with finite nonempty support, rational coordinates and maximum at most 1 in modules. We also set

$$\mathbf{Q}_s = \{(x_1, f_1, \dots, x_n, f_n) : x_i, f_i \in \mathbf{Q}, i = 1, \dots, n \\ \text{range}(x_i) \cup \text{range}(f_i) < \text{range}(x_{i+1}) \cup \text{range}(f_{i+1}) \forall i < n\}.$$

We consider a coding function σ (i.e. σ is an injection) from \mathbf{Q}_s to the set $\{2j : j \in \mathbb{N}\}$ such that for every $\phi = (x_1, f_1, \dots, x_n, f_n) \in \mathbf{Q}_s$

$$(2.1) \quad \sigma(x_1, f_1, \dots, x_{n-1}, f_{n-1}) < \sigma(x_1, f_1, \dots, x_n, f_n)$$

$$(2.2) \quad \max\{\text{range}(x_n) \cup \text{range}(f_n)\} \leq m_{\sigma(\phi)}^{\frac{1}{2}}$$

Although x_i, f_i are elements of c_{00} their role in the space X_{ius} we shall define is quite different. Namely x_i will be elements of the space itself and f_i elements of its dual X_{ius}^* . For similar reasons we shall denote the standard basis of c_{00} either by $(e_n)_n$ or $(e_n^*)_n$.

Definition 2.1. A sequence $\phi = (x_1, f_1, \dots, x_{2k}, f_{2k}) \in \mathbf{Q}_s$ is said to be a **special sequence of length $2k$** provided that

$$(2.3) \quad x_1 = \frac{1}{n_{2j}} \sum_{l=1}^{n_{2j}} e_{1,l}, \quad f_1 = \frac{1}{m_{2j}} \sum_{l=1}^{n_{2j}} e_{1,l}^*, \text{ for some } j \in \mathbb{N}, \text{ such that } m_{2j}^{1/2} > 2k,$$

where $(e_{1,l})_{l=1}^{n_{2j}}$ is a subset of the standard basis of c_{00} of cardinality n_{2j} , and for every $1 \leq i \leq k$, setting $\phi_i = (x_1, f_1, \dots, x_i, f_i)$

$$(2.4) \quad \|f_{2i}\|_\infty \leq \frac{1}{m_{\sigma(\phi_{i-1})}}, \quad |f_{2i}(x_{2i})| \leq \frac{1}{m_{\sigma(\phi_{i-1})}},$$

$$(2.5) \quad \text{if } i < k \text{ then } x_{2i+1} = \frac{1}{n_{\sigma(\phi_{2i})}} \sum_{l=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,l}, \quad f_{2i+1} = \frac{1}{m_{\sigma(\phi_{2i})}} \sum_{l=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,l}^*,$$

where for every $i \geq 1$, $(e_{2i+1,l})_{l=1}^{n_{\sigma(\phi_{2i})}}$ is a subset of the standard basis of c_{00} of cardinality $n_{\sigma(\phi_{2i})}$.

The norming set of the space X_{ius} .

The norming set K will be equal to the union $\cup_{n=0}^{\infty} K_n$ and the sequence $(K_n)_n$ is increasing and inductively defined. The inductive definition of K_n goes as follows:

We set

$$K_0^0 = K_0 = \{\pm e_n^* : n \in \mathbb{N}\} \text{ and } K_0^j = \emptyset \text{ for } j = 1, 2, \dots$$

Assume that $K_{n-1} = \cup_j K_{n-1}^j$ has been defined. Then we set,

(a) for $j \in \mathbb{N}$

$$K_n^{2j} = K_{n-1}^{2j} \cup \left\{ \frac{1}{m_{2j}} \sum_{i=1}^d f_i : d \leq n_{2j}, f_1 < \dots < f_d, f_i \in K_{n-1} \right\}.$$

(b) For $j \in \mathbb{N}$ and every $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ special sequence of length n_{2j+1} , (see Definition 2.1), such that $f_{2i} \in K_{n-1}^{\sigma(\phi_{2i-1})}$ for $i = 1, \dots, n_{2j+1}/2$ (where $\phi_{2i-1} = (x_1, f_1, \dots, x_{2i-1}, f_{2i-1})$) we define the set

$$(2.6) \quad K_{n,\phi}^{2j+1} = \left\{ \frac{\pm 1}{m_{2j+1}} E(\lambda_{f'_2} f_1 + f'_2 + \dots + \lambda_{f'_{n_{2j+1}}}} f_{n_{2j+1}-1} + f'_{n_{2j+1}}) : \right.$$

$$(2.7) \quad \left. E \text{ interval of } \mathbb{N}, \text{ supp } f'_{2i} = \text{supp } f_{2i}, f'_{2i} \in K_{n-1}^{\sigma(\phi_{2i-1})}, \right.$$

$$\left. |g(x_{2i})| \leq \frac{1}{m_{\sigma(\phi_{2i-1})}} \text{ for all } g \in K_{n-1}^{\sigma(\phi_{2i-1})} \right.$$

$$\left. \lambda_{f'_{2i}} = f'_{2i}(m_{\sigma(\phi_{2i-1})} x_{2i}) \text{ if } f'_{2i}(x_{2i}) \neq 0, \frac{\pm 1}{n_{2j+1}^2} \text{ otherwise} \right\}.$$

We define

$$K_n^{2j+1} = \cup \{K_{n,\phi}^{2j+1} : \phi \text{ is a special sequence of length } n_{2j+1}\} \cup K_{n-1}^{2j+1},$$

and finally we set

$$K_n = \cup_j K_n^j.$$

This completes the inductive definition of K_n and we set,

$$K = \cup_n K_n.$$

Let us observe that the set K satisfies the following properties

- (i) It is symmetric and for each $f \in K$, $\|f\|_{\infty} \leq 1$.
- (ii) It is closed under interval projections (i.e. it is closed in the restriction of its elements on intervals).
- (iii) It is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operations (i.e. for $f_1 < f_2 < \dots < f_d$ in K with $d \leq n_{2j}$ we have that $\frac{1}{m_{2j}} \sum_{l=1}^d f_l \in K$).
- (iv) If $f \in K$ then either $f = \pm e_n^*$ or $f \in K_n^j$ for $n \geq 1, j \in \mathbb{N}$. In the later case we define the **weight** of f as $w(f) = m_j$. Note that $w(f)$ is not necessarily unique.

The space X_{ius} is the completion of the space $(c_{00}, \|\cdot\|_K)$ where

$$\|x\|_K = \sup \{ \langle f, x \rangle : f \in K \}.$$

From the definition of the norming set K it follows easily that $(e_n)_n$ is a bimonotone basis of X_{ius} . Also it is easy to see, using (iii), that the basis $(e_n)_n$ is boundedly complete.

Indeed, for $x \in c_{00}$ and $E_1 < E_2 < \dots < E_{n_{2j}}$ intervals of \mathbb{N} it follows from property (iii) of the norming set that,

$$\|x\| \geq \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|.$$

Also from the choice of the sequences $(n_i)_i, (m_i)_i$ it follows that $\frac{n_{2j}}{m_{2j}}$ increases to infinity. These observations easily yield that the basis is boundedly complete.

To prove that the space X_{ius} is reflexive we need to show that the basis is shrinking. This requires some further work and we will present the argument later.

Lemma 2.2. *Let $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ be a special sequence of length n_{2j+1} such that:*

- (a) $\{f_i : i = 1, \dots, n_{2j+1}\} \subset K$ and for $i \geq 2$, $w(f_i) = m_{\sigma(\phi_{i-1})}$.
- (b) For $1 \leq i \leq n_{2j+1}/2$, $\|w(f_{2i})x_{2i}\| \leq 1$.

Then there exists $n \in \mathbb{N}$ such that $K_{n,\phi}^{2j+1}$ is nonempty.

Notation. For every ϕ special sequence of length n_{2j+1} such that $K_{n,\phi}^{2j+1} \neq \emptyset$ for some n we define $K_\phi = \cup_n K_{n,\phi}^{2j+1}$.

Remark 2.3. Let us point out that in the definition of the special sequences we have attempted to connect averages of the basis with block vectors that are quite freely chosen. This will be used to show that the quotient map from the space to the space $X_{ius}/\langle e_n \rangle_{n \in M}$ is a strictly singular operator. Moreover we keep the dependence only between f_{2i-1} and the family $\{g \in K : w(g) = w(f_{2i}), \text{supp}(g) = \text{supp}(f_{2i})\}$ to ensure that the space X_{ius} is unconditionally saturated.

Definition 2.4 (The tree \mathcal{T}_f of a functional $f \in K$). Let $f \in K$. We call *tree* of f (or tree corresponding to the analysis of f) every finite family $\mathcal{T}_f = (f_\alpha)_{\alpha \in \mathcal{A}}$ indexed by a finite tree \mathcal{A} with a unique root $0 \in \mathcal{A}$ such that the following conditions are satisfied:

- 1) $f_0 = f$ and $f_\alpha \in K$ for each $\alpha \in \mathcal{A}$.
- 2) If $\alpha \in \mathcal{A}$ is terminal node then $f_\alpha \in K_0$.
- 3) For every $\alpha \in \mathcal{A}$ which is not terminal, denoting by S_α the set of the immediate successors of α , exclusively one of the following two holds:

- (a) $S_\alpha = \{\beta_1, \dots, \beta_d\}$ with $f_{\beta_1} < \dots < f_{\beta_d}$ and there exists $j \in \mathbb{N}$ such that $d \leq n_{2j}$,

$$\text{and } f_\alpha = \frac{1}{m_{2j}} \sum_{i=1}^d f_{\beta_i}.$$

- (b) There exists a special sequence $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ of length n_{2j+1} , an

$$\text{interval } E \text{ and } \varepsilon \in \{-1, 1\} \text{ such that } f_\alpha = \frac{\varepsilon}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} E(\lambda_{f'_{2i}} f_{2i-1} + f'_{2i}) \in K_\phi$$

$$\text{and } \{f_\beta : \beta \in S_\alpha\} = \{E f_{2i-1} : E f_{2i-1} \neq 0\} \cup \{E f'_{2i} : E f'_{2i} \neq 0\}.$$

It follows from the inductive definition of K that every $f \in K$ admits a tree, not necessarily unique.

3. THE SPACE X_{ius} IS UNCONDITIONALLY SATURATED

This section is devoted to show that the space X_{ius} is unconditionally saturated. We start with the following: We set

$$\tilde{K} = \{\pm e_n, \frac{1}{m_{2j}} \sum_{i \in F} \pm e_i : \#F \leq n_{2j}, j \in \mathbb{N}\} \cup \{0\}.$$

Clearly \tilde{K} is a subset of the norming set K and it is easily checked that \tilde{K} is a countable and compact set (in the pointwise topology). It is well known that the space $C(\tilde{K})$ is c_0 -saturated. Observe also that $\|\cdot\|_{\tilde{K}} \leq \|\cdot\|_{X_{ius}}$ and hence the identity operator

$$I : (c_{00}, \|\cdot\|_{X_{ius}}) \rightarrow (c_{00}, \|\cdot\|_{\tilde{K}})$$

is bounded. Since the basis $(e_n)_n$ of X_{ius} is boundedly complete, the space X_{ius} does not contain c_0 , therefore the operator I is also strictly singular. These observations yield that every block subspace Y of X_{ius} contains a further block sequence (y_n) such that $\|y_n\|_{X_{ius}} = 1$ and $\|y_n\|_{\tilde{K}} \xrightarrow{n} 0$. Our intention is to show the following:

Proposition 3.1. *Let $(x_\ell)_\ell$ be a normalized block sequence in X_{ius} such that $\|x_\ell\|_{\tilde{K}} \rightarrow 0$. Then there exists a subsequence $(x_\ell)_{\ell \in M}$ of (x_ℓ) which is an unconditional basic sequence.*

The proof of this proposition requires certain steps and we attempt a sketch of the main ideas. First we assume, passing to a subsequence, that $\|x_\ell\|_{\tilde{K}} < \sigma_\ell$ with $\sum \sigma_\ell < \frac{1}{8}$ and we claim that $(x_\ell)_{\ell \in \mathbb{N}}$ is an unconditional basic sequence. Indeed, consider a norm one combination $\sum_{\ell=1}^d b_\ell x_\ell$ and let $(\varepsilon_\ell)_{\ell=1}^d \in \{-1, 1\}^d$. We shall show that $\|\sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell\| > \frac{1}{4}$. Choose any $f \in K$ with $f(\sum_{\ell=1}^d b_\ell x_\ell) > \frac{3}{4}$ and we are seeking a $g \in K$ such that $g(\sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell) \geq \frac{1}{4}$. To find such a g a normal procedure is to consider a tree $(f_\alpha)_{\alpha \in \mathcal{A}}$ of the functional f and then inductively to produce a functional g with a tree $(g_\alpha)_{\alpha \in \mathcal{A}}$ such that

$$(3.1) \quad |f(x_\ell) - g(\varepsilon_\ell x_\ell)| < 2\sigma_\ell$$

which easily yields the desired result.

In most of the cases, the choice for producing g_α from f_α is straightforward. Essentially there exists only one case where we need to be careful. That is when $f_\alpha \in K_\phi$ for some special sequence ϕ . (i.e. $f_\alpha = \frac{\pm 1}{m_{2j+1}} E(\lambda_{f'_2} f_1 + f'_2 + \dots + \lambda_{f'_{n_{2j+1}-1}} f_{n_{2j+1}-1} + f_{n_{2j+1}})$) and for some $i \leq n_{2j+1}/2$ and $\ell < d$ we have

$$\max \supp x_{\ell-1} < \min \supp (f_{2i-1}) \leq \max \supp x_\ell$$

$$\max \supp f'_{2i} \geq \min \supp x_{\ell+1}.$$

In this case we produce g_α from f_α such that $g_\alpha \in K_\phi$. The form of f_α and hence g_α permits us to show that $|f_\alpha(x_\ell) - g_\alpha(\varepsilon_\ell x_\ell)| < 2\sigma_\ell$.

We pass now to present the proof and we start with the next notation and definitions.

Notation. Let $f \in K$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ a tree of f . Then for every non terminal node $\alpha \in \mathcal{A}$ we order the set S_α following the natural order of $\{\supp f_\beta\}_{\beta \in S_\alpha}$. Hence for $\beta \in S_\alpha$ we denote by β^+ the immediate successor of β in the above order if such an object exists.

Definition 3.2. Let $f \in K$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . A couple of functionals f_α, f_{α^+} is said to be a **depended couple with respect to f** , (w.r.t. f), if there exists $\beta \in \mathcal{A}$ such that $\alpha, \alpha^+ \in S_\beta$, $f_\beta = \frac{\varepsilon}{m_{2j+1}} E(\sum_{i=1}^{n_{2j+1}/2} \lambda_{f_{2i}^\beta} f_{2i-1}^\beta + f_{2i}^\beta)$, $f_\alpha = E f_{2i-1}^\beta$ and $f_{\alpha^+} = E f_{2i}^\beta$ for some $i \leq n_{2j+1}/2$.

Definition 3.3. Let $(x_k)_k$ be a normalized block sequence, $f \in K$ and $\mathcal{T}_f = (f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . For $k \in \mathbb{N}$, a couple of functionals f_α, f_{α^+} is said to be **depended couple with respect to f and \mathbf{x}_k** (w.r.t.) if f_α, f_{α^+} is a depended couple w.r.t. f and moreover

$$\max \text{supp } x_{k-1} < \min \text{supp } f_\alpha \leq \max \text{supp } x_k$$

$$\text{and } \max \text{supp } f_{\alpha^+} \geq \min \text{supp } x_{k+1}.$$

We also set

$$(3.2) \quad \mathcal{F}_{f, x_k} = \{\alpha \in \mathcal{A} : f_\alpha, f_{\alpha^+} \text{ is a depended couple w.r.t. } f \text{ and } x_k\}.$$

and

$$(3.3) \quad \mathcal{F}_f = \bigcup_k \mathcal{F}_{f, x_k}.$$

Remark 3.4. Let $(x_k)_k$ be a block sequence in X_{ius} , $f \in K$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f .

1. It is easy to see that for every $k \in \mathbb{N}$ and every non terminal node $\alpha \in \mathcal{A}$ the set $S_\alpha \cap \mathcal{F}_{f, x_k}$ has at most one element.

2. As consequence of this, we obtain that for every k and $\alpha_1, \alpha_2 \in \mathcal{F}_{f, x_k}$ with $\alpha_1 \neq \alpha_2$ we have that α_1, α_2 are incomparable and $|\alpha_1| \neq |\alpha_2|$, where we denote by $|\alpha|$ the order of α as a member of the finite tree \mathcal{A} .

3. It is also easy to see that for $\alpha_1, \alpha_2 \in \mathcal{F}_f$ with $\alpha_1 \neq \alpha_2$, α_1, α_2 are incomparable and hence $\text{range}(f_{\alpha_1}) \cap \text{range}(f_{\alpha_2}) = \emptyset$.

Lemma 3.5. Let $(x_k)_k$ be a block sequence in X_{ius} such that $\|x_k\|_{\tilde{K}} \leq \sigma_k$, $f \in K$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . We set $y_k = x_k|_{\cup_{\alpha \in \mathcal{F}_f} \text{supp}(f_\alpha)}$. Then we have that

$$(3.4) \quad |f(y_k)| \leq 2\sigma_k.$$

Proof. Let us first observe that for each $q \in \mathbb{N}$ the set $\{\text{range}(f_\alpha) : |\alpha| = q\}$ consists of pairwise disjoint sets. Therefore from the preceding remark we obtain that for each k and each q the set

$$\{\alpha \in \mathcal{F}_f : |\alpha| = q, \text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset\}$$

contains at most two elements (one of them belongs to \mathcal{F}_{f, x_k} and the other to \mathcal{F}_{f, x_ℓ} for some $\ell \leq k-1$). Therefore

$$\begin{aligned} |f(y_k)| &\leq \sum_{\alpha \in \mathcal{F}_f} \left(\prod_{0 \leq \gamma \prec \alpha} \frac{1}{w(f_\gamma)} \right) |f_\alpha(x_k)| \\ &= \sum_i \sum_{\alpha \in \mathcal{F}_f, |\alpha|=i} \left(\prod_{0 \leq \gamma \prec \alpha} \frac{1}{w(f_\gamma)} \right) |f_\alpha(x_k)| \leq 2\sigma_k \sum_i \frac{1}{m_1^i} \leq 2\sigma_k. \end{aligned}$$

□

The following lemma is the crucial step for the proof of the main result of this section.

Lemma 3.6. Let $(x_k)_k$ be a block sequence in X_{ius} , $f \in K$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . For every $k \in \mathbb{N}$ we set $y_k = x_k|_{\cup_{\alpha \in \mathcal{F}_f} \text{supp}(f_\alpha)}$. Then for every choice of signs $(\varepsilon_k)_k$ there exists a functional $g \in K$ with a tree $(g_\alpha)_{\alpha \in \mathcal{A}}$ such that

- (1) $f(x_k - y_k) = g(\varepsilon_k(x_k - y_k))$
- (2) For every $\alpha \in \mathcal{A}$, $\text{supp}(f_\alpha) = \text{supp}(g_\alpha)$
- (3) $\mathcal{F}_{f, x_k} = \mathcal{F}_{g, x_k}$

for every $k = 1, 2, \dots$

Proof. For the given tree $(f_\alpha)_{\alpha \in \mathcal{A}}$ of f , we define

$$D = \{\beta \in \mathcal{A} : \text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset \text{ for at most one } k \\ \text{and if } \beta \in S_\alpha \text{ then } \text{range}(f_\alpha) \cap \text{range}(x_i) \neq \emptyset \text{ for at least two } x_i\}.$$

Let us observe that for every branch b of \mathcal{A} , $b \cap D$ is a singleton. Furthermore, for $\beta \in D$ and $\gamma \in \mathcal{A}$ with $\beta \prec \gamma$ we have that $\gamma \notin \mathcal{F}_f$.

The definition of $(g_\alpha)_{\alpha \in \mathcal{A}}$ requires the following three steps.

Step 1. First we define the set $\{g_\beta : \beta \in D\}$ as follows.

(a) If $\beta \in D$ and there exists $\alpha \in \mathcal{A}$ with $\alpha \preceq \beta$ and f_α, f_{α^+} is a depended couple w.r.t. f we set $g_\beta = f_\beta$.

(b) If $\beta \in D$ does not belong to the previous case and there exists a (unique) k such that $\text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset$ then we set $g_\beta = \varepsilon_k f_\beta$.

(c) If $\beta \in D$ does not belong to case (a) and $\text{range}(f_\beta) \cap \text{range}(x_k) = \emptyset$ for all k then we set $g_\beta = \varepsilon_k f_\beta$ where

$$k = \max\{l : \text{range}(x_l) < \text{range}(f_\beta)\}.$$

(We have assumed that $\min \text{range}(x_1) \leq \min \text{range}(f)$.)

Let us comment the case (a) in the above definition. First we observe that the unique $\alpha \in \mathcal{A}$ witnessing that β belongs to the case (a) satisfies the following: either $\alpha = \beta$ or $|\alpha| = |\beta| - 1$. Moreover if this α does not belong to \mathcal{F}_f then $\alpha = \beta$, $\alpha^+ \in D$. In this case, if we assume that there exists a (unique) k such that $\text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset$ then g_{α^+} is defined by cases (b) or (c) and $g_{\alpha^+} = \varepsilon_k f_{\alpha^+}$ for the specific k . All these are straightforward consequences of the corresponding definitions.

Step 2. We set

$$D^+ = \{\gamma \in \mathcal{A} : \text{there exists } \beta \in D \text{ with } \beta \prec \gamma\}.$$

For $\gamma \in D^+$ we set $g_\gamma = \varepsilon_\beta f_\gamma$ where β is the unique element of D with $\beta \prec \gamma$ and $\varepsilon_\beta \in \{-1, 1\}$ is such that $g_\beta = \varepsilon_\beta f_\beta$.

Clearly for every $\beta \in D \cup D^+$, $(g_\gamma)_{\beta \preceq \gamma}$ is a tree of the functional g_β . Furthermore for $\alpha \in D \cup D^+$ the following properties hold:

- (1) $\text{supp}(f_\alpha) = \text{supp}(g_\alpha)$
- (2) $w(f_\alpha) = w(g_\alpha)$

Step 3. We set

$$D^- = \{\alpha \in \mathcal{A} : \text{there exists } \beta \in D \text{ with } \alpha \prec \beta\}.$$

Observe that $\mathcal{A} = D \cup D^+ \cup D^-$ and using backward induction, for all $\alpha \in D^-$ we shall define g_α such that the above (1) and (2) hold and additionally the following two properties will be established.

- (3) For $\alpha \in D^-$, $f_\alpha(x_k - y_k) = g_\alpha(\varepsilon_k(x_k - y_k))$ for all k .
- (4) For $\alpha \in D^-$ and each k we have that $\mathcal{F}_{f_\alpha, x_k} = \mathcal{F}_{g_\alpha, x_k}$.

Observe that for every $\alpha \in D^-$ we have that $f_\alpha \notin K_0$ and furthermore for every $\beta \in D$ $\mathcal{F}_{f_\beta} = \emptyset$.

We pass now to construct inductively g_α , $\alpha \in D^-$ and to establish properties (1)–(4). Let assume that $\alpha \in D^-$ and for every $\beta \in S_\alpha$ either $\beta \in D$ or g_β has been defined and properties (1)–(4) have been established. We consider the following three cases.

Case 1. $w(f_\alpha) = m_{2j}$ and $\alpha \in \mathcal{F}_f$.

That means that $f_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} f_\beta$ and each $f_\beta = e_\ell^*$ for some $\ell \in \mathbb{N}$. Then $S_\alpha \subset D$ and

from Step 1(a) we conclude that $g_\beta = f_\beta$ for all $\beta \in S_\alpha$. We set

$$g_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} g_\beta = f_\alpha.$$

Furthermore for each k we have that $\text{supp}(g_\alpha) \cap \text{supp}(x_k) \subset \text{supp}(y_k)$. Hence

$$g_\alpha(\varepsilon_k(x_k - y_k)) = f_\alpha(x_k - y_k) = 0$$

and also $\mathcal{F}_{g_\alpha} = \mathcal{F}_{f_\alpha} = \emptyset$. Thus properties (3) and (4) hold while (1) and (2) are obvious.

Before passing to the next case let us notice that there is no $\alpha \in D^-$ such that $f_\alpha, f_{\alpha+}$ is a depended couple w.r.t. f and $\alpha \notin \mathcal{F}_f$. (See the comments after Step 1.)

Case 2. $w(f_\alpha) = m_{2j}$ and $\alpha \notin \mathcal{F}_f$.

From the previous observation we obtain that $\alpha \neq \beta$ for each $\beta \in \mathcal{A}$ with $f_\beta, f_{\beta+}$ depended couple w.r.t. f , and we set

$$g_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} g_\beta.$$

Our inductive assumptions yield properties (1) and (2). To establish property (3) let $k \in \mathbb{N}$ and $\beta \in D \cap S_\alpha$ be such that $\text{range}(x_k) \cap \text{range}(f_\beta) \neq \emptyset$. Then $g_\beta = \varepsilon_k f_\beta$ hence

$$g_\beta(\varepsilon_k(x_k - y_k)) = \varepsilon_k g_\beta(x_k - y_k) = f_\beta(x_k - y_k).$$

If $\beta \in D^- \cap S_\alpha$ by the inductive assumption for each k we have

$$g_\beta(\varepsilon_k(x_k - y_k)) = f_\beta(x_k - y_k).$$

Therefore

$$g_\alpha(\varepsilon_k(x_k - y_k)) = f_\alpha(x_k - y_k).$$

Finally, for each k

$$\mathcal{F}_{f_\alpha, x_k} = \bigcup_{\beta \in S_\alpha} \mathcal{F}_{f_\beta, x_k} = \bigcup_{\beta \in S_\alpha \cap D^-} \mathcal{F}_{f_\beta, x_k} = \bigcup_{\beta \in S_\alpha \cap D^-} \mathcal{F}_{g_\beta, x_k} = \mathcal{F}_{g_\alpha, x_k}$$

which establishes property (4).

Case 3. $f_\alpha = \frac{\varepsilon}{m_{2j+1}} E(\lambda_{f_2^\alpha} f_1^\alpha + f_2^\alpha + \dots + \lambda_{f_{n_{2j+1}}^\alpha} f_{n_{2j+1}-1}^\alpha + f_{n_{2j+1}}^\alpha) \in K_\phi$ where $\{f_\beta : \beta \in S_\alpha\} = \{E f_i^\alpha : E f_i^\alpha \neq 0, 1 \leq i \leq n_{2j+1}\}$, $\varepsilon \in \{-1, 1\}$, E is an interval and ϕ is a special sequence of length n_{2j+1} .

Let $\phi = (z_1, f_1, \dots, z_{n_{2j+1}}, f_{n_{2j+1}})$. Without loss of generality we assume that $E = \mathbb{N}$ and $\varepsilon = 1$. Let us observe that the definition of $\{g_\beta : \beta \in D\}$ and the inductive assumptions yield that for $i \leq n_{2j+1}/2$,

- (i) $f_{2i-1} = f_{2i-1}^\alpha = g_{2i-1}^\alpha$.
- (ii) $w(f_{2i}) = w(f_{2i}^\alpha) = w(g_{2i}^\alpha)$.
- (iii) $\text{supp}(f_{2i}) = \text{supp}(f_{2i}^\alpha) = \text{supp}(g_{2i}^\alpha)$.

We define

$$g_\alpha = \frac{1}{m_{2j+1}} \left(\lambda_{g_2^\alpha} f_1 + g_2^\alpha + \lambda_{g_4^\alpha} f_3 + g_4^\alpha + \dots + \lambda_{g_{n_{2j+1}}^\alpha} f_{n_{2j+1}-1} + g_{n_{2j+1}}^\alpha \right)$$

where $\{g_\beta : \beta \in S_\alpha\} = \{g_i^\alpha : 1 \leq i \leq n_{2j+1}\}$ while $\lambda_{g_{2i}^\alpha}$ are defined as follows:

(5) If $g_{2i}^\alpha(z_{2i}) \neq 0$ then $\lambda_{g_{2i}^\alpha} = g_{2i}^\alpha(m_{\sigma(\phi_{2i-1})} z_{2i})$.

(6) If $g_{2i}^\alpha(z_{2i}) = 0$ and $f_{2i-1}^\alpha = f_\beta$, there are two cases

- a) If $\beta \in \mathcal{F}_f$ or $\beta \notin \mathcal{F}_f$ and $\text{range}(f_\beta) \cap \text{range}(x_k) = \emptyset$ for all k we set $\lambda_{g_{2i}^\alpha} = \frac{1}{n_{2j+1}}$.
- b) If $\beta \notin \mathcal{F}_f$ and there exists (unique) k such that $\text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset$ then we set $\lambda_{g_{2i}^\alpha} = \varepsilon_k \lambda_{f_{2i}^\alpha}$.

Let us observe that in the case (6) b), as follows from the comments after Step 1, $g_{\beta^+} = \varepsilon_k f_{\beta^+}$ hence $f_{\beta^+}(z_{2i}) = 0$ if and only if $g_{\beta^+}(z_{2i}) = 0$.

From the above definition of $\lambda_{g_{2i}^\alpha}$, $1 \leq i \leq n_{2j+1}/2$ and (i),(ii),(iii), we obtain that the functional g_α belongs to $K_\phi \subset K$.

Properties (1) and (2) are obvious for g_α and we check the rest. First we establish property (4).

Let k be given. From Remark 3.4 (1) it follows that there exists at most one depended couple $f_{2i-1}^\alpha, f_{2i}^\alpha$ w.r.t. f and x_k . Moreover if such a depended couple, $f_{2i-1}^\alpha, f_{2i}^\alpha$, exists then for every $i' \neq i$ it holds that $\mathcal{F}_{f_{2i'}, x_k}^\alpha = \emptyset$. Therefore in this case we have that

$$(3.5) \quad \mathcal{F}_{f_\alpha, x_k} = \mathcal{F}_{f_{2i}^\alpha, x_k} \cup \{\beta\}$$

where $f_{2i-1}^\alpha = f_\beta$. In the case that no such depended couple exists, it follows that $\mathcal{F}_{f_{2i}^\alpha, x_k}^\alpha \neq \emptyset$ for at most one i . This is a consequence of the definitions and the fact that the functionals $(f_i^\alpha)_i$ are successive. If such an i exists then

$$(3.6) \quad \mathcal{F}_{f_\alpha, x_k} = \mathcal{F}_{f_{2i}^\alpha, x_k}$$

The last alternative is that $\mathcal{F}_{f_\alpha, x_k} = \emptyset$. This description of $\mathcal{F}_{f_\alpha, x_k}$ and the inductive assumptions easily yield property (4). Namely, either $\mathcal{F}_{g_\alpha, x_k} = \mathcal{F}_{f_{2i}^\alpha, x_k} \cup \{\beta\}$ if (3.5) holds, $\mathcal{F}_{g_\alpha, x_k} = \mathcal{F}_{f_{2i}^\alpha, x_k}$ if (3.6) holds, or $\mathcal{F}_{g_\alpha, x_k} = \emptyset$.

Finally we check property (3). Fix a number k and $i \leq n_{2j+1}/2$. If $g_{2i}^\alpha = g_\beta$ and $\beta \in D^-$ the inductive assumption provides

$$(3.7) \quad g_{2i}^\alpha(\varepsilon_k(x_k - y_k)) = f_{2i}^\alpha(x_k - y_k).$$

If $\beta \in D$ and $\text{range}(f_{2i}^\alpha) \cap \text{range}(x_k) \neq \emptyset$ then $g_{2i}^\alpha = \varepsilon_k f_{2i}^\alpha$ which yields (3.7). Also if $\text{range}(f_{2i}^\alpha) \cap \text{range}(x_k) = \emptyset$ equality (3.7) trivially holds.

In the case $g_{2i-1}^\alpha = g_\beta$, $\beta \in S_\alpha$ we distinguish two subcases. First assume that $\beta \in \mathcal{F}_f$. Then $\text{supp}(g_{2i-1}^\alpha) = \text{supp}(f_{2i-1}^\alpha)$ and $\text{supp}(f_{2i-1}^\alpha) \cap \text{supp}(x_k - y_k) = \emptyset$ therefore

$$g_{2i-1}^\alpha(\varepsilon_k(x_k - y_k)) = 0 = f_{2i-1}^\alpha(x_k - y_k).$$

The second subcase is $\beta \notin \mathcal{F}_f$. As we have explained in the comments after Step 1 that means that either $\text{range}(f_\beta) \cap \text{range}(x_k) = \emptyset$, hence everything trivially holds, or $\beta, \beta^+ \in D$, $g_{\beta^+} = \varepsilon_k f_{\beta^+}$ and $\lambda_{g_{2i}^\alpha} = \varepsilon_k \lambda_{f_{2i}^\alpha}$. From these observations we conclude that

$$\lambda_{g_{2i}^\alpha} g_{2i-1}^\alpha(\varepsilon_k(x_k - y_k)) = \lambda_{f_{2i}^\alpha} f_{2i-1}^\alpha(x_k - y_k).$$

All these derive the desired equality, namely

$$g_\alpha(\varepsilon_k(x_k - y_k)) = f_\alpha(x_k - y_k).$$

The inductive construction and the entire proof of the lemma is complete. \square

Proof of Proposition 3.1. Let $(\sigma_\ell)_\ell$ be a decreasing sequence of positive numbers such that $\sum_\ell \sigma_\ell \leq 1/8$. For each $\ell \in \mathbb{N}$ we select k_ℓ such that $\|x_{k_\ell}\|_{\tilde{K}} < \sigma_\ell$. For simplicity we assume that the entire sequence (x_ℓ) satisfies the above condition. Let $\sum_{\ell=1}^d b_\ell x_\ell$ be a finite linear combination which maximizes the norm of all vectors of the form $\sum_{\ell=1}^d c_\ell x_\ell$ with $|c_\ell| = |b_\ell|$. Assume furthermore that $\|\sum_{\ell=1}^d b_\ell x_\ell\| = 1$ and let $f \in K$ with $f(\sum_{\ell=1}^d b_\ell x_\ell) \geq 3/4$. Choose $\{\varepsilon_\ell\}_{\ell=1}^d \in \{-1, 1\}^d$ and consider the vector $\sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell$. Lemma 3.6 yields that there exists $g \in K$ and that for each $\ell = 1, \dots, d$, there exists a vector y_ℓ such that

$$(3.8) \quad g\left(\sum_{\ell=1}^d \varepsilon_\ell b_\ell (x_\ell - y_\ell)\right) = f\left(\sum_{\ell=1}^d b_\ell (x_\ell - y_\ell)\right).$$

Also Lemma 3.5 and Lemma 3.6(2) and (3) yield that

$$|g(y_\ell)| \leq 2\sigma_\ell \quad \text{and} \quad |f(y_\ell)| \leq 2\sigma_\ell \quad \text{for all } \ell = 1, \dots, d.$$

Hence

$$\begin{aligned} \left\| \sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell \right\| &\geq \left| g\left(\sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell\right) \right| \geq \left| g\left(\sum_{\ell=1}^d \varepsilon_\ell b_\ell (x_\ell - y_\ell)\right) \right| - \sum_{\ell=1}^d |g(y_\ell)| \\ &\geq \left| f\left(\sum_{\ell=1}^d b_\ell x_\ell\right) \right| - \sum_{\ell=1}^d |g(y_\ell)| - \sum_{\ell=1}^d |f(y_\ell)| \geq 3/4 - 2/4 = 1/4. \end{aligned}$$

This completes the proof of the proposition. \square

4. THE SPACE X_{ius} IS INDECOMPOSABLE

In the last section we shall show that the space X_{ius} is indecomposable. This will be a consequence of a stronger result concerning the structure of the space $\mathcal{B}(X_{ius})$ of the bounded linear operators acting on X_{ius} . The proof adapts techniques related to H.I. spaces as they were presented in [AT]. Thus we will first consider the auxiliary space X_u and we will estimate the norm of certain averages of its basis. Next we will use the basic inequality to reduce upper estimation on certain averages to the previous results. Finally we shall compute the norms of linear combinations related to special sequences.

The auxiliary spaces $X_u, X_{u,k}$

We begin with the definition of the space X_u which will be used to provide us upper estimations for certain averages in the space X_{ius} .

The space X_u is the mixed Tsirelson space $T[(\mathcal{A}_{4n_j}, \frac{1}{m_j})_{j=1}^\infty]$. The norming set W of X_u is defined in a similar manner as the set K .

We set $W_0^j = \{\pm e_n^* : n \in \mathbb{N}\} \cup \{0\}$, for $j \in \mathbb{N}$, $W_0 = \cup_j W_0^j$. In the general inductive step we define

$$W_n^j = W_{n-1}^j \cup \left\{ \frac{1}{m_j} \sum_{i=1}^d f_i : d \leq 4n_j, f_1 < \dots < f_d \in W_{n-1} \right\}$$

and $W_n = \cup_j W_n^j$. Finally let $W = \cup_n W_n$. The space X_u is the completion of $(c_{00}, \|\cdot\|_W)$ where

$$\|x\|_W = \sup\{\langle f, x \rangle : f \in W\}.$$

It is clear that the norming set K of the space X_{ius} is a subset of the convex hull of W . Hence we have that $\|x\|_K \leq \|x\|_W$ for every $x \in c_{00}$.

We also need the spaces $X_{u,k} = T[(\mathcal{A}_{4n_j}, \frac{1}{m_j})_{j=1}^k]$. The norm of such a space is denoted by $\|\cdot\|_{u,k}$ and it is defined in a similar manner as the norm of X_u . Namely we define W_n^j , $n \in \mathbb{N}$, $1 \leq j \leq k$ as above and $W_n^{(k)} = \bigcup_{j=1}^k W_n^j$. The norming set is $W^{(k)} = \bigcup_{n=0}^\infty W_n^{(k)}$.

Spaces of this form have been studied in [BD] and it has been shown that such a space is either isomorphic to some ℓ_p , $1 < p < \infty$, or to c_0 .

Before stating the next lemma we introduce some notations. For each $k \in \mathbb{N}$ we set $q_k = \frac{1}{\log_{4n_k} m_k}$ and $p_k = \frac{1}{1 - \log_{4n_k} m_k}$ its conjugate.

Lemma 4.1. *For the sequences $(m_j)_j, (n_j)_j$ used in the definition of X_{ius} and $X_u, X_{u,k}$ the following hold:*

- (1) *The sequence $(q_j)_j$ strictly increases to infinity.*

(2) For $x = \sum a_\ell e_\ell \in c_{00}$, $\|x\|_{u,k} \leq \|x\|_{p_k}$.

(3) $\|\frac{1}{n_{k+1}} \sum_{i=1}^{n_{k+1}} e_i\|_{p_k} \leq \frac{1}{m_{k+1}^3}$.

Proof. (1) Using that $m_{j+1} = m_j^5$ and $n_{j+1} = (4n_j)^{s_j}$ and the fact that s_j increases to infinity we have that

$$q_{j+1} = \frac{1}{\log_{4n_{j+1}} m_{j+1}} = \frac{1}{\log_{4(4n_j)^{s_j}} m_j^5} > \frac{1}{\frac{5}{s_j} \log_{4n_j} m_j} = \frac{s_j}{5} q_j$$

hence $(q_j)_j$ strictly increases to infinity.

(2) We inductively show that for $f \in W_n^{(k)}$

$$|f(\sum a_\ell e_\ell)| \leq \|\sum a_\ell e_\ell\|_{p_k}.$$

For $n = 0$ it is trivial. The general inductive step goes as follows: for $f \in W_{n+1}^{(k)}$

$$f(\sum a_\ell e_\ell) = \frac{1}{m_j} \sum_{i=1}^d f_i(\sum a_\ell e_\ell)$$

where $f_1 < f_2 < \dots < f_d$, $d \leq 4n_j$ for some $j \leq k$. We set $E_i = \text{range}(f_i)$ and from our inductive assumption and Hölder inequality we obtain that

$$|f(\sum a_\ell e_\ell)| \leq \frac{1}{m_j} \sum_{i=1}^d \|\sum_{\ell \in E_i} a_\ell e_\ell\|_{p_k} \leq \frac{d^{\frac{1}{q_j}}}{m_j} \left(\sum_{i=1}^d \|\sum_{\ell \in E_i} a_\ell e_\ell\|_{p_k}^{p_j} \right)^{\frac{1}{p_j}}.$$

Using that $p_k \leq p_j$ and $m_j = (4n_j)^{\frac{1}{q_j}}$ we obtain inequality (2).

(3)

$$\|\frac{1}{n_{k+1}} \sum_{i=1}^{n_{k+1}} e_i\|_{p_k} \leq \frac{1}{n_{k+1}^{\frac{1}{q_k}}} = \frac{1}{(4n_k)^{\frac{s_k}{q_k}}} = \frac{1}{m_k^{s_k}} \leq \frac{1}{m_{k+1}^3}.$$

(Recall that $2^{s_k} \geq m_{k+1}^3$).

□

The tree \mathcal{T}_f of $f \in W$ is defined in a similar manner as for $f \in K$.

Lemma 4.2. *Let $f \in W$ and $j \in \mathbb{N}$. Then*

$$(4.1) \quad |f(\frac{1}{n_j} \sum_{i=1}^{n_j} e_{k_i})| \leq \begin{cases} \frac{2}{w(f) \cdot m_j}, & \text{if } w(f) < m_j \\ \frac{1}{w(f)}, & \text{if } w(f) \geq m_j. \end{cases}$$

If moreover we assume that there exists a tree $(f_\alpha)_{\alpha \in \mathcal{A}}$ of f , such that $w(f_\alpha) \neq m_j$ for every $\alpha \in \mathcal{A}$, we have that

$$(4.2) \quad |f(\frac{1}{n_j} \sum_{i=1}^{n_j} e_{k_i})| \leq \frac{2}{m_j^3}.$$

In particular the above upper estimations holds for every $f \in K$.

Proof. If $w(f) \geq m_j$ the estimation is an immediate consequence of the fact that $\|f\|_\infty \leq 1/w(f)$. Let $w(f) < m_j$ and $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . We set

$$B = \{i : \text{there exists } \alpha \in \mathcal{A} \text{ with } k_i \in \text{supp } f_\alpha \text{ and } w(f_\alpha) \geq m_j\}$$

Then we have that

$$(4.3) \quad |f(\frac{1}{n_j} \sum_{i \in B} e_{k_i})| \leq \frac{1}{w(f)m_j}.$$

To estimate $|f(\frac{1}{n_j} \sum_{i \in B^c} e_{k_i})|$, we observe that $f|_{\{k_i: i \in B^c\}} \in W^{(j-1)}$ (the norming set of $X_{u,j-1}$) hence Lemma 4.1 yields that

$$(4.4) \quad |f(\frac{1}{n_j} \sum_{i \in B^c} e_{k_i})| \leq \frac{1}{m_j^3}.$$

Combining (4.3) and (4.4) we obtain (4.1).

To see (4.2) we define the set

$$B = \{i : \text{there exists } \alpha \in \mathcal{A} \text{ with } k_i \in \text{supp } f_\alpha \text{ and } w(f_\alpha) \geq m_{j+1}\}$$

and we conclude that

$$(4.5) \quad |f(\frac{1}{n_j} \sum_{i \in B} e_{k_i})| \leq \frac{1}{m_{j+1}} < \frac{1}{m_j^3}.$$

Furthermore from our assumption $w(f_\alpha) \neq m_j$ for every $\alpha \in \mathcal{A}$ we conclude that $f|_{\{k_i: i \in B^c\}} \in W^{(j-1)}$. This yields that the corresponding of (4.4) remains valid and combining (4.4) and (4.5) we obtain (4.2). \square

The basic inequality and its consequences

Next we state and prove the basic inequality which is an adaptation of the corresponding result from [AT]. Actually the proof of the present statement is easier than the original one, due mainly to the low complexity of the family \mathcal{A}_n (in [AT] are studied spaces defined with use of the Schreier families $(\mathcal{S}_\xi)_{\xi < \omega_1}$) and also since the definition of the norming set K does not involve convex combinations. The role of this result is important since it includes most of the necessary computations (unconditional or conditional).

Recall that K and W denote the norming sets of X_{ius} and X_u respectively.

Proposition 4.3. (*Basic inequality*) Let (x_k) be a block sequence in X_{ius} , (j_k) be a strictly increasing sequence of positive integers, $(b_k) \in c_{00}$, $C \geq 1$ and $\varepsilon > 0$ such that

a) $\|x_k\| \leq C$ for every k .

b) For every $k \geq 1$, $\#(\text{supp } x_k) \frac{1}{m_{j_k+1}} \leq \varepsilon$.

c) For every $k \geq 1$, for all $f \in K$ with $w(f) < m_{j_k}$, we have that $|f(x_k)| \leq \frac{C}{w(f)}$.

Then for every $f \in K$ there exists g_1 such that $g_1 = h_1$ or $g_1 = e_t^* + h_1$ where $t \notin \text{supp } h_1$, $h_1 \in W$, $w(h_1) = w(f)$, and $g_2 \in c_{00}$ with $\|g_2\|_\infty \leq \varepsilon$ such that

$$(4.6) \quad |f(\sum b_k x_k)| \leq C(g_1 + g_2)(\sum |b_k| e_k),$$

and $\text{supp } g_1, \text{supp } g_2$ are contained in $\{k : \text{supp}(f) \cap \text{range}(x_k) \neq \emptyset\}$.

d) If we additionally assume that for some $j_0 \in \mathbb{N}$ we have that

$$(4.7) \quad |f(\sum_{k \in E} b_k x_k)| \leq C(\max_{k \in E} |b_k| + \varepsilon \sum_{k \in E} |b_k|),$$

for every interval E of positive integers and every $f \in K$ with $w(f) = m_{j_0}$, then h_1 may be selected to have a tree $(h_\alpha)_{\alpha \in \mathcal{A}_1}$ such that $w(h_\alpha) \neq m_{j_0}$ for every $\alpha \in \mathcal{A}_1$.

Our intention is to apply the above inequality in order to obtain upper estimations for ℓ_1 -averages of rapidly increasing sequences. Observe that the above proposition reduces this problem to the estimations of the functionals g_1, g_2 on a corresponding average of the basis in the space X_u .

The proof in the general case, assuming only a), b), c), and in the special case, where additionally d) is assumed, is the same. We will make the proof only in the special case.

The proof in the general case arises by omitting any reference to the question whether a functional has weight m_{j_0} or not. For the rest of the proof we assume that there exists $j_0 \in \mathbb{N}$ such that condition d) in the statement of Proposition is fulfilled.

Proof of Proposition 4.3. Let $f \in K$ and let $\mathcal{T}_f = (f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree of f . For every k such that $\text{supp}(f) \cap \text{range}(x_k) \neq \emptyset$ we define the set A_k as follows:

$$A_k = \left\{ \alpha \in \mathcal{A} : \begin{aligned} &(i) \text{ } \text{supp} f_\alpha \cap \text{range}(x_k) = \text{supp}(f) \cap \text{range}(x_k), \\ &(ii) \text{ for all } \gamma \prec \alpha, \text{ } w(f_\gamma) \neq m_{j_0}, \\ &(iii) \text{ there is no } \beta \in S_\alpha \text{ such that} \\ &\quad \text{supp}(f_\alpha) \cap \text{range}(x_k) = \text{supp}(f_\beta) \cap \text{range}(x_k) \text{ if } w(f_\alpha) \neq m_{j_0} \end{aligned} \right\}.$$

From the definition, it follows easily that for every k such that $\text{supp}(f) \cap \text{range}(x_k) \neq \emptyset$ A_k is a singleton.

We recursively define sets $(D_\alpha)_{\alpha \in \mathcal{A}}$ as follows.

For every terminal node α of the tree we set $D_\alpha = \{k : \alpha \in A_k\}$. For every non terminal node α we define,

$$D_\alpha = \{k : \alpha \in A_k\} \cup \bigcup_{\beta \in S_\alpha} D_\beta.$$

The following are easy consequences of the definition.

- i) If $\beta \prec \alpha$, $D_\alpha \subset D_\beta$.
- ii) If $w(f_\alpha) = m_{j_0}$, then $D_\beta = \emptyset$ for all $\beta \succ \alpha$.
- iii) If $w(f_\alpha) \neq m_{j_0}$, then for every $\{k\} : k \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta\} \cup \{D_\beta : \beta \in S_\alpha\}$ is a family of successive subsets of \mathbb{N} .
- iv) If $w(f_\alpha) \neq m_{j_0}$, for every $k \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta$ there exists $\beta \in S_\alpha$ such that $\min \text{supp} x_k < \min \text{supp} f_\beta \leq \max \text{supp} x_k$ and for $k' \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta$ different from k the corresponding β' is different from β .

Inductively for every $\alpha \in \mathcal{A}$ we define g_α^1 and g_α^2 such that

- (1) For every $\alpha \in \mathcal{A}$, $\text{supp} g_\alpha^1$ and $\text{supp} g_\alpha^2 \subset D_\alpha$.
- (2) If $w(f_\alpha) = m_{j_0}$, $g_\alpha^1 = e_{k_\alpha}^*$, where $|b_{k_\alpha}| = \max_{k \in D_\alpha} |b_k|$ and $g_\alpha^2 = \varepsilon \sum_{k \in D_\alpha} e_k^*$.
- (3) If $w(f_\alpha) \neq m_{j_0}$, $g_\alpha^1 = h_\alpha$ or $g_\alpha^1 = e_{k_\alpha}^* + h_\alpha$ where $k_\alpha \notin \text{supp} h_\alpha$, $h_\alpha \in W$ and $w(h_\alpha) = w(f_\alpha)$.
- (4) For every $\alpha \in \mathcal{A}$ the following inequality holds

$$|f_\alpha(\sum_{k \in D_\alpha} b_k x_k)| \leq C(g_\alpha^1 + g_\alpha^2)(\sum_{k \in D_\alpha} |b_k| e_k).$$

For every terminal node we set $g_\alpha^1 = g_\alpha^2 = 0$ if $D_\alpha = \emptyset$, otherwise we set $g_\alpha = e_k^*$ if $D_\alpha = \{k\}$ and $g_\alpha^2 = 0$. Assume that we have defined the functionals g_β^1 and g_β^2 , satisfying (1) – (4), for every $\beta \in \mathcal{A}$ with $|\beta| = k$, and let $\alpha \in \mathcal{A}$ with $|\alpha| = k - 1$. If $D_\alpha = \emptyset$ we set $g_\alpha^1 = g_\alpha^2 = 0$. Let $D_\alpha \neq \emptyset$. We distinguish two cases.

Case 1. $w(f_\alpha) = m_j \neq m_{j_0}$.

Let $T_\alpha = D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta = \{k : \alpha \in A_k\}$. We set $T_\alpha^2 = \{k \in T_\alpha : m_{j_{k+1}} \leq m_j\}$ and $T_\alpha^1 = T_\alpha \setminus T_\alpha^2$. In the pointwise estimations we shall make below, we shall discard the coefficient $\lambda_{f_{2i}}$, which appears in the definition of the special functionals, since $|\lambda_{f_{2i}}| \leq 1$.

From condition b) in the statement, it follows that for each $k \in T_\alpha^2$

$$(4.8) \quad |f_\alpha(x_k)| \leq \#(\text{supp} x_k) \|f_\alpha\|_\infty \leq \#(\text{supp} x_k) \frac{1}{m_j} \leq \varepsilon \leq C\varepsilon.$$

We define

$$g_\alpha^2 = \varepsilon \sum_{k \in T_\alpha^2} e_k^* + \sum_{\beta \in S_\alpha} g_\beta^2.$$

We observe that $\|g_\alpha^2\|_\infty \leq \varepsilon$, and that $|f_\alpha(x_k)| \leq C\varepsilon = Cg_\alpha^2(e_k)$, for every $k \in T_\alpha^2$.

Let $T_\alpha^1 = \{k_1 < k_2 < \dots < k_l\}$. By the definition of T_α^1 we have that $m_j < m_{j_{k_2}} < m_{j_{k_3}} < \dots < m_{j_{k_l}}$. Thus condition c) in the statement implies that

$$(4.9) \quad |f_\alpha(x_{k_i})| \leq \frac{C}{m_j} = \frac{1}{m_j} e_{k_i}^*(Ce_{k_i}), \quad \text{for every } 2 \leq i \leq l.$$

We set

$$g_\alpha^1 = e_{k_1}^* + \frac{1}{m_j} \left(\sum_{i=2}^l e_{k_i}^* + \sum_{\beta \in S_\alpha} g_\beta^1 \right).$$

(The term $e_{k_1}^*$ does not appear if $w(f_\alpha) < m_{j_k}$ for every $k \in T_\alpha$). We have to show that $h_\alpha = \frac{1}{m_j} (\sum_{i=2}^l e_{k_i}^* + \sum_{\beta \in S_\alpha} g_\beta^1) \in W$. From the inductive hypothesis, we have that $g_\beta^1 = h_\beta$ or $g_\beta^1 = e_{k_\beta}^* + h_\beta$, $h_\beta \in W$, for every $\beta \in S_\alpha$. For $\beta \in S_\alpha$, such that $g_\beta^1 = e_{k_\beta}^* + h_\beta$, let $E_\beta^1 = \{n \in \mathbb{N} : n < k_\beta\}$ and $E_\beta^2 = \{n \in \mathbb{N} : n > k_\beta\}$. We set $h_\beta^1 = E_\beta^1 h_\beta$, $h_\beta^2 = E_\beta^2 h_\beta$. For every β such that $g_\beta^1 = e_{k_\beta}^* + h_\beta$, the functionals h_β^1 , $e_{k_\beta}^*$, h_β^2 are successive belonging to W , and for $\beta \neq \beta' \in S_\alpha$ the corresponding functionals have disjoint range, since $\text{supp} g_\beta^1$ is an interval, remark (iii) after the definition of D_α . From the remark iv) after the definition of D_α we have that $\#T_\alpha^1 \leq n_j$. It follows that

$$\#(\{e_{k_i}^*, 2 \leq i \leq l\} \cup \{e_{k_\beta}^*, h_\beta^1, h_\beta^2 : \beta \in S_\alpha, g_\beta = e_{k_\beta}^* + h_\beta\} \cup \{h_\beta : \beta \in S_\alpha, g_\beta = h_\beta\}) \leq 4n_j.$$

Therefore $h_\alpha = \frac{1}{m_j} (\sum_{i=2}^l e_{k_i}^* + \sum_{\beta \in S_\alpha} g_\beta^1) \in W$. It remains to show property 4). By (4.9) we have that $|f_\alpha(x_{k_i})| \leq Cg_\alpha^1(e_{k_i})$ for every $2 \leq i \leq l$, while

$$|f_\alpha(x_{k_1})| \leq \|x_{k_1}\| \leq Ce_{k_1}^*(e_{k_1}) = g_\alpha^1(Ce_{k_1}).$$

We also have that

$$\begin{aligned} |f_\alpha(\sum_{k \in \cup_{\beta \in S_\alpha} D_\beta} b_k x_k)| &\leq \sum_{\beta \in S_\alpha} |f_\alpha(\sum_{k \in D_\beta} b_k x_k)| \\ &\leq \frac{1}{m_j} \sum_{\beta \in S_\alpha} |f_\beta(\sum_{k \in D_\beta} b_k x_k)| \\ &\leq \frac{1}{m_j} \sum_{\beta \in S_\alpha} (g_\beta^1 + g_\beta^2) (C \sum_{k \in D_\beta} |b_k| e_k) \\ &\leq (g_\alpha^1 + g_\alpha^2) (C \sum_{k \in D_\alpha} |b_k| e_k). \end{aligned}$$

Case 2. $w(f_\alpha) = m_{j_0}$. In this case we have that D_α is an interval of the positive integers and $D_\gamma = \emptyset$, for every $\gamma \succ \alpha$. Let k_α such that $b_{k_\alpha} = \max_{k \in D_\alpha} |b_k|$. We set

$$g_\alpha^1 = e_{k_\alpha}^* \quad \text{and} \quad g_\alpha^2 = \varepsilon \sum_{k \in D_\alpha} e_k^*.$$

Then we have that

$$|f_\alpha(\sum_{k \in D_\alpha} b_k x_k)| \leq C(\max_{k \in D_\alpha} |b_k| + \varepsilon \sum_{k \in D_\alpha} |b_k|) = (g_\alpha^1 + g_\alpha^2) (C \sum_{k \in D_\alpha} |b_k| e_k).$$

□

Definition 4.4. Let $k \in \mathbb{N}$. A vector $x \in c_{00}$ is said to be a $C - \ell_1^k$ average if there exists $x_1 < \dots < x_k$, $\|x_i\| \leq C\|x\|$ and $x = \frac{1}{k} \sum_{i=1}^k x_i$. Moreover, if $\|x\| = 1$ then x is called a normalized $C - \ell_1^k$ average.

Lemma 4.5. Let $j \geq 1$, x be an $C - \ell_1^{n_j}$ -average. Then for every $n \leq n_{j-1}$ and every $E_1 < \dots < E_n$, we have that

$$\sum_{i=1}^n \|E_i x\| \leq C(1 + \frac{2n}{n_j}) < \frac{3}{2}C.$$

We refer to [S], (or [GM], Lemma 4), for a proof.

Proposition 4.6. For every normalized block sequence $(y_\ell)_\ell$ and every $k \geq m_2$ there exists a linear combination of $(y_\ell)_\ell$ which is a normalized $2 - \ell_1^k$ average.

Proof. Given $k \geq m_2$ there exists $j \in \mathbb{N}$ such that $m_{2j-1} < k \leq m_{2j+1}$. Recall that $n_{2j+2} = (4n_{2j+1})^{s_{2j+1}}$ and $m_{2j+2}^3 < 2^{s_{2j+1}}$. Hence setting $s = s_{2j+1}$ we have that $k^s \leq n_{2j+2}$ and $2^{-s} < \frac{1}{m_{2j+2}}$. Observe that

$$(4.10) \quad \left\| \sum_{i=1}^{k^s} y_i \right\| \geq \frac{k^s}{m_{2j+2}}.$$

Assuming that there is no normalized $2 - \ell_1^k$ average in $\langle y_i : i \leq k^s \rangle$ and following the proof of Lemma 3 in [GM] we obtain that

$$(4.11) \quad \left\| \sum_{i=1}^{k^s} y_i \right\| < k^s \cdot 2^{-s}.$$

Since $2^{-s} < \frac{1}{m_{2j+2}}$, (4.10) and (4.11) derive a contradiction. \square

Definition 4.7. A block sequence (x_k) in X_{ius} is said to be a (C, ε) -rapidly increasing sequence (R.I.S.), if there, exists a strictly increasing sequence (j_k) of positive integers such that

- a) $\|x_k\| \leq C$.
- b) $\#(\text{range}(x_k)) \frac{1}{m_{j_{k+1}}} < \varepsilon$.
- c) For every $k = 1, 2, \dots$ and every $f \in K$ with $w(f) < m_{j_k}$ we have that $|f(x_k)| \leq \frac{C}{w(f)}$.

Remark 4.8. Let $(x_k)_k$ be a block sequence in X_{ius} such that each x_k is a normalized $\frac{2C}{3} - \ell_1^{n_{j_k}}$ average and let $\varepsilon > 0$ be such that for each k , $\#(\text{range}(x_k)) \frac{1}{m_{j_{k+1}}} < \varepsilon$. Then Lemma 4.5 yields that condition (c) in the above definition is also satisfied hence $(x_k)_k$ is a (C, ε) R.I.S. In this case we shall call $(x_k)_k$ as a **(C, ε) R.I.S. of ℓ_1 averages**. Let also observe that Proposition 4.6 ensures that for every block sequence $(y_\ell)_\ell$ and every $\varepsilon > 0$ there exists $(x_k)_k$ which is a $(3, \varepsilon)$ R.I.S. of ℓ_1 averages.

Proposition 4.9. Let $(x_k)_{i=1}^{n_j}$ be a (C, ε) -R.I.S such that $\varepsilon \leq \frac{1}{n_j}$. Then

1) For every $f \in K$

$$\left| f\left(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k\right) \right| \leq \begin{cases} \frac{3C}{m_j w(f)}, & \text{if } w(f) < m_j \\ \frac{C}{w(f)} + \frac{2C}{n_j}, & \text{if } w(f) \geq m_j. \end{cases}$$

In particular $\left\| \frac{1}{n_j} \sum_{k=1}^{n_j} x_k \right\| \leq \frac{2C}{m_j}$.

2) If for $j_0 = j$ the assumption d) of the basic inequality is fulfilled (Proposition 4.3), for a linear combination $\frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i$, where $|b_i| \leq 1$, then

$$\left\| \frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i \right\| \leq \frac{4C}{m_j^3}.$$

3) If $(x_i)_{i=1}^{n_{2j}}$ is a $(3, \varepsilon)$ rapidly increasing sequence of ℓ_1 averages then

$$(4.12) \quad \frac{1}{m_{2j}} \leq \left\| \frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i \right\| \leq \frac{6}{m_{2j}}.$$

Proof. The proof of 1) is an application of the basic inequality and Lemma 4.2. Indeed for $f \in K$, the basic inequality yields that there exist $h_1 \in W$ with $w(f) = w(h_1)$, $t \in \mathbb{N}$ with $t \notin \text{supp} h_1$, and $h_2 \in c_{00}$ with $\|h_2\|_\infty \leq \varepsilon$, such that

$$(4.13) \quad \left| f\left(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k\right) \right| \leq (e_t^* + h_1 + h_2)C \left(\frac{1}{n_j} \sum_{k=1}^{n_j} e_k\right).$$

Using Lemma 4.2 and the fact that $\varepsilon \leq \frac{1}{n_j}$ we obtain

$$(4.14) \quad \left| f\left(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k\right) \right| \leq \begin{cases} \frac{C}{n_j} + \frac{2C}{w(f)m_j} + C\varepsilon \leq \frac{3C}{w(f)m_j} & \text{if } w(f) < m_j \\ \frac{C}{n_j} + \frac{C}{w(f)} + C\varepsilon \leq \frac{C}{w(f)} + \frac{2C}{n_j} & \text{if } w(f) \geq m_j. \end{cases}$$

To prove 2) we observe that the basic inequality yields the existence of h_1, h_2 such that h_1 has a tree $(h_\alpha)_{\alpha \in \mathcal{A}}$ such that $w(h_\alpha) \neq m_j$ for every $\alpha \in \mathcal{A}$ and $\|h_2\|_\infty \leq \varepsilon$. This and Lemma 4.2 yield that

$$(4.15) \quad \left| f\left(\frac{1}{n_j} \sum_{k=1}^{n_j} b_k x_k\right) \right| \leq (e_t^* + h_1 + h_2)C \left(\frac{1}{n_j} \sum_{k=1}^{n_j} e_k\right) \leq \frac{C}{n_j} + \frac{2C}{m_j^3} + C\varepsilon \leq \frac{4C}{m_j^3}.$$

The upper estimation in 3) follows from 1) for $C = 3$. For the lower estimation in 3), for every $i \leq n_{2j}$ we choose a functional f_i belonging to the pointwise closure of K such that $f_i(x_i) = 1$ and $\text{range}(f_i) \subset \text{range}(x_i)$. Then it is easy to see that the functional $f = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i$ belongs to the same set and provides the required result. \square

Proposition 4.10. *The space X_{ius} is reflexive.*

Proof. As we have already explained after the definition of the norming set K , the basis is boundedly complete. Therefore to show that the space X_{ius} is reflexive we need to prove that the basis is shrinking.

Assume on the contrary. Namely there exists $x^* = w^* - \sum_{n=1}^{\infty} b_n e_n^*$ and $x^* \notin \overline{\langle e_n^* \rangle}$.

Then there exists $\varepsilon > 0$ and successive intervals $(E_k)_k$ such that $\|E_k x^*\| > \varepsilon$. Choose $(x_k)_k$ in X_{ius} such that $\text{supp}(x_k) \subset E_k$, $\|x_k\| = 1$ and $x^*(x_k) > \varepsilon$. It follows that every convex combination $\sum a_k x_k$ satisfies

$$(4.16) \quad \left\| \sum a_k x_k \right\| > \varepsilon.$$

Next for j sufficiently large such that $\frac{4}{\varepsilon m_{2j}} < \varepsilon$ we define $y_1, y_2, \dots, y_{n_{2j}}$ a $(\frac{2}{\varepsilon}, \frac{1}{n_{2j}})$ R.I.S. of ℓ_1 averages and each y_i is some average of $(x_k)_k$. Proposition 4.9 (1) yields that

$$(4.17) \quad \left\| \frac{1}{n_{2j}} (y_1 + y_2 + \dots + y_{n_{2j}}) \right\| \leq \frac{4}{m_{2j}\varepsilon} < \varepsilon.$$

Clearly (4.17) contradicts (4.16) and the basis is shrinking. \square

The structure of $\mathcal{B}(X_{ius})$

Definition 4.11. A sequence $\chi = (x_1, f_1, x_2, f_2, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ is said to be a **de-pended sequence of length n_{2j+1}** if the following conditions are fulfilled

- (i) There exists $\phi = (x_1, f_1, y_2, f_2, \dots, x_{2i-1}, f_{2i-1}, y_{2i}, f_{2i}, \dots, y_{n_{2j+1}}, f_{n_{2j+1}})$ special sequence of length n_{2j+1} such that $\text{supp}(y_{2i}) = \text{supp}(x_{2i})$ and $\|y_{2i} - x_{2i}\| \leq \frac{1}{n_{2i}^2}$ where for $1 \leq i < n_{2j+1}$, $j_{i+1} = \sigma(\phi_i)$.
- (ii) For $i \leq n_{2j+1}/2$ we have that

$$x_{2i} = \frac{c_{2i}}{n_{j_{2i}}} \sum_{l=1}^{n_{j_{2i}}} x_l^{2i}$$

where $(x_l^{2i})_l$ is a $(3, \frac{1}{n_{j_{2i}}})$ R.I.S. of ℓ^1 averages, $c_{2i} \in (0, 1)$.

- (iii) $f_{2i}(x_{2i}) \geq \frac{1}{12m_{j_{2i}}}$.

The following is a consequence of the previous results, and we sketch the proof of it.

Lemma 4.12. Let $(y_k)_k$ be a normalized block sequence in X_{ius} and $(e_n)_{n \in M}$ be a subsequence of its basis. Then for all $j \in \mathbb{N}$ there exists a depended sequence

$$\chi = (x_1, f_1, x_2, f_2, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$$

of length n_{2j+1} such that for each $i \leq n_{2j+1}/2$, $x_{2i-1} \in \langle e_n \rangle_M$ and $x_{2i} \in \langle y_k \rangle_k$.

Proof. Let $j_1 \in \mathbb{N}$, j_1 even such that $m_{j_1}^{1/2} > n_{2j+1}$. We set

$$x_1 = \frac{1}{n_{j_1}} \sum_{i=1}^{n_{j_1}} e_{1,i} \text{ and } f_1 = \frac{1}{m_{j_1}} \sum_{i=1}^{n_{j_1}} e_{1,i}^*,$$

such that $x_1 \in \langle e_n \rangle_M$. Let $j_2 = \sigma(x_1, f_1)$. Using Proposition 4.6 we choose an $(3, \frac{1}{n_{j_2}})$ R.I.S, $(x_l^2)_{l=1}^{n_{j_2}} \in \langle y_k \rangle_k$ such that $x_1 < x_l^2$ for every $l \leq n_{j_2}$. Next we choose for every $l \leq n_{j_2}$ a functional $f_l^2 \in K$ such that $f_l^2(x_l^2) \geq \frac{2}{3}\|x_l^2\| \geq \frac{2}{3}$ and $\text{range}(f_l^2) \subset \text{range}(x_l^2)$. We set

$$f_2 = \frac{1}{m_{j_2}} \sum_{l=1}^{n_{j_2}} f_l^2 \text{ and } x_2 = \frac{c_2}{n_{j_2}} \sum_{l=1}^{n_{j_2}} x_l^2 \text{ where } c_2 = \frac{1}{6}(1 - \frac{m_{j_2}}{n_{j_2}^2}).$$

From Proposition 4.9, it follows that $\|x_2\| \leq (\frac{1}{m_{j_2}} - \frac{1}{n_{j_2}^2})$. We also have that

$$(4.18) \quad f_2(x_2) \geq \frac{1}{m_{j_2}} \frac{c_2}{n_{j_2}} \sum_{l=1}^{n_{j_2}} f_l^2(x_l^2) \geq \frac{2}{3} \frac{c_2}{m_{j_2}} \geq \frac{1}{12m_{j_2}}.$$

We choose $y_2 \in \mathbf{Q}$, that is y_2 is a finite sequence with rational coordinates, such that $\|y_2 - x_2\| \leq \frac{1}{n_{j_2}^2}$ and $\text{supp}(y_2) = \text{supp}(x_2)$. It follows that $\|y_2\| \leq \frac{1}{m_{j_2}}$ and therefore (x_1, f_1, y_2, f_2) is a special sequence of length 2.

We set $j_3 = \sigma(x_1, f_1, y_2, f_2)$ and we choose

$$x_3 = \frac{1}{n_{j_3}} \sum_{l=1}^{n_{j_3}} e_{3,l} \text{ and } f_3 = \frac{1}{m_{j_3}} \sum_{l=1}^{n_{j_3}} e_{3,l}^*,$$

such that $\text{range}(y_2) \cup \text{range}(f_2) < \text{range}(x_3)$ and $x_3 \in \langle e_n \rangle_M$. Next we choose x_4, f_4 and y_4 as in the second step, and it is clear that the procedure goes through up to the choice of $x_{n_{2j+1}}, f_{n_{2j+1}}$ and $y_{n_{2j+1}}$. \square

Remark 4.13. a) Let us observe that the proof of Lemma 4.12 yields that if $\chi = (x_1, f_1, x_2, f_2, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ is a depended sequence, then for every $i \leq n_{2j+1}/2$ it holds that $x_{2i} = \frac{c_{2i}}{n_{j_{2i}}} \sum_{l=1}^{n_{j_{2i}}} x_l^{2i}$, where $(x_l^{2i})_l$ is a $(3, n_{j_{2i}}) - R.I.S.$, $j_{2i} = \sigma(\phi_{2i-1})$ and $c_{2i} \leq \frac{1}{6}$. It follows from Proposition 4.9 that

$$\|m_{j_{2i}} x_{2i}\| \leq 1, \text{ and also if } f \in K \text{ and } w(f) < m_{j_{2i}} \text{ then, } f(m_{j_{2i}} x_{2i}) \leq \frac{2}{w(f)}.$$

b) Definition 4.11 essentially describes that a depended sequence is a small perturbation of a special sequence. Its necessity occurs from the restriction in the definition of the special sequence $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ that each $x_i \in \mathbf{Q}$ (i.e. $x_i(n)$ is a rational number) not permitting to find such elements x_i in every block subspace.

Next we state the basic estimations of averages related to depended sequences.

Lemma 4.14. *Let $\chi = (x_1, f_1, x_2, f_2, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ be a depended sequence of length n_{2j+1} . Then the following inequality holds:*

$$\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i \right\| \leq \frac{8}{m_{2j+1}^3}$$

where $m_{j_i} = w(f_i)$.

Lemma 4.15. *Let $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ be a special sequence. For every $i \leq n_{2j+1}/2$, let $\sigma(x_1, f_1, \dots, x_{2i-1}, f_{2i-1}) = j_{2i}$ and let $y_{2i} = \frac{m_{j_{2i}}}{n_{j_{2i}}} \sum_{l=1}^{n_{j_{2i}}} e_{k_l}$ be such that*

$$\text{supp}(f_{2i}) \cap \text{supp}(y_{2i}) = \emptyset \text{ and } \text{supp}(f_{2i-1}) < \text{supp}(y_{2i}) < \text{supp}(f_{2i+1}).$$

Then it holds that

$$\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right\| \leq \frac{8}{m_{2j+1}^3}.$$

These two lemmas are the key ingredients for proving the main results for the structure of X_{ius} and $\mathcal{B}(X_{ius})$. We proceed with the proof of the main results and we will provide the proof of the two lemmas at the end.

Proposition 4.16. *Let $M \in [\mathbb{N}]$ and let $(y_k)_k$ be a normalized block sequence. Then we have that*

$$\text{dist}(S_{\langle e_n \rangle_M}, S_{\langle y_k \rangle_k}) = 0.$$

Proof. For a given $\varepsilon > 0$ we choose $j \in \mathbb{N}$ such that $\frac{8}{m_{2j+1}^3} < \varepsilon$. From Lemma 4.12 there exists a depended sequence $\chi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ such that $x_{2i-1} \in \langle e_n \rangle_M$, $x_{2i} \in \langle y_k \rangle_k$ for every $i \leq n_{2j+1}/2$. Set

$$e = \frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_{j_{2i-1}} x_{2i-1} \text{ and } y = \frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_{j_{2i}} x_{2i}.$$

We have that $e \in \langle e_n : n \in M \rangle$ and $y \in \langle y_i : i \in M \rangle$. From Lemma 4.14 we have that $\|e - y\| \leq \frac{8}{m_{2j+1}^3}$. To obtain a lower estimation of the norm of e and y we consider the functional $f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \lambda_{f_{2i}} f_{2i-1} + f_{2i}$ where $\lambda_{f_{2i}} = f_{2i}(m_{j_{2i}} y_{2i})$ and

$\phi = (x_1, f_1, y_2, f_2, \dots, y_{n_{2j+1}}, f_{n_{2j+1}})$ is the special sequence associated to the depended sequence χ . From the definition of the depended sequence, $f_{2i}(m_{j_{2i}}x_{2i}) \geq \frac{1}{12}$, and $\|x_{2i} - y_{2i}\| \leq \frac{1}{n_{j_{2i}}^2}$ for every $i \leq n_{2j+1}/2$. It follows that

$$\lambda_{f_{2i}} = f(m_{j_{2i}}y_{2i}) \geq f(m_{j_{2i}}x_{2i}) - m_{j_{2i}}\|x_{2i} - y_{2i}\| > \frac{1}{12} - \frac{1}{m_{j_{2i}}^2} > \frac{1}{24}.$$

Therefore

$$(4.19) \quad \|e\| \geq f(e) = \frac{m_{2j+1}}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \frac{\lambda_{f_{2i}} f_{2i-1}(m_{j_{2i-1}}x_{2i-1})}{n_{2j+1}} \geq \frac{1}{48},$$

and

$$(4.20) \quad \|y\| \geq f(y) = \frac{m_{2j+1}}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \frac{f_{2i}(m_{j_{2i}}x_{2i})}{n_{2j+1}} \geq \frac{1}{24}.$$

These lower estimations and the fact that $\|e - y\| \leq \frac{8}{m_{2j+1}^2}$ easily yields the desired result. \square

Lemma 4.17. *Let $T : X_{ius} \rightarrow X_{ius}$ be a bounded operator. Then*

$$\lim_n \text{dist}(Te_n, \mathbb{R}e_n) = 0.$$

Proof. Without loss of generality we may assume that $\|T\| = 1$. Since (e_n) is weakly null, by a small perturbation of T we may assume that $T(e_n)$ is a finite block, $T(e_n) \in \mathbf{Q}$ and $\min \text{supp} T(e_n) \xrightarrow{n} \infty$. Let $I(e_n)$ be the smallest interval containing $\text{supp} T(e_n) \cup \text{supp}(e_n)$. Passing to a subsequence $(e_n)_{n \in M}$, we may assume that $I(e_n) < I(e_m)$ for every $n, m \in M$ with $n < m$.

If the result is not true, we may assume, on passing to a further subsequence, that there exists $\delta > 0$ such that

$$\text{dist}(Te_n, \mathbb{R}e_n) > 2\delta \quad \text{for every } n \in M.$$

It follows that $\|P_{n-1}Te_n\| > \delta$ or $\|(I - P_n)Te_n\| > \delta$. Therefore for every $n \in M$ we can choose $x_n^* \in K$ such that

$$(4.21) \quad x_n^*(Te_n) \geq \delta, \quad \text{range}(x_n^*) \cap \text{range}(e_n) = \emptyset, \quad \text{and} \quad \text{range}(x_n^*) \subset I(e_n).$$

Since T is bounded, for every $j \in \mathbb{N}$ we have that

$$\|T(\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} e_{k_i})\| \leq \|T\| \|\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} e_{k_i}\| = \frac{1}{m_{2j}}.$$

Also for every $j \in \mathbb{N}$ and $k_1 < k_2 < \dots < k_{n_{2j}}$ in M , the functional $h_{2j} = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} x_{k_i}^*$ is in K and

$$\|T(\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} e_{k_i})\| = \|\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} Te_{k_i}\| \geq h_{2j}(\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} Te_{k_i}) \geq \frac{\delta}{m_{2j}}.$$

We consider now a *special sequence* $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ which is defined as follows: for every $i \geq 0$,

$$\begin{aligned} x_{2i+1} &= \frac{1}{n_{\sigma(\phi_{2i})}} \sum_{j=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,j}, & f_{2i+1} &= \frac{1}{m_{\sigma(\phi_{2i})}} \sum_{j=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,j}^* \\ x_{2i} &= \frac{1}{n_{\sigma(\phi_{2i-1})}} \sum_{j=1}^{n_{\sigma(\phi_{2i-1})}} T e_{2i,j}, & f_{2i} &= \frac{1}{m_{\sigma(\phi_{2i-1})}} \sum_{j=1}^{n_{\sigma(\phi_{2i-1})}} x_{2i,j}^* \end{aligned}$$

where $e_{i,\ell} \in \{e_n : n \in M\}$, $x_{2i,j}^*$, $T e_{2i,j}$ satisfies (4.21), and $I(e_{i,\ell}) < I(e_{s,j})$ if either $i < s$ or $i = s$ and $\ell < j$. This is possible by our assumption $I(e_n) < I(e_m)$ for $n, m \in M$ with $n < m$. Observe that $f_{2i}(m_{\sigma(\phi_{2i-1})} x_{2i}) \geq \delta$ and also that $\text{range}(f_\ell) \cap \text{range}(x_{2i}) = \emptyset$ for every $\ell \neq 2i$. Consider now the following vector:

$$x = \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \frac{m_{\sigma(\phi_{2i-1})}}{n_{\sigma(\phi_{2i-1})}} \sum_{j=1}^{n_{\sigma(\phi_{2i-1})}} e_{2i,j}.$$

Then

$$T(x) = \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_{\sigma(\phi_{2i-1})} x_{2i},$$

and

$$\|Tx\| \geq \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} (\lambda_{f_{2i}} f_{2i-1} + f_{2i})Tx \geq \frac{\delta}{2m_{2j+1}}.$$

On the other hand, if $y_{2i} = \frac{m_{\sigma(\phi_{2i-1})}}{n_{\sigma(\phi_{2i-1})}} \sum_{j=1}^{n_{\sigma(\phi_{2i-1})}} e_{2i,j}$, then we have that $\text{supp}(y_{2i}) \cap \text{supp} f_{2i} = \emptyset$ and $x_{2i-1} < y_{2i} < x_{2i+1}$ for every $i \leq n_{2j+1}/2$, and therefore by Lemma 4.15 we have that

$$\|x\| = \left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right\| \leq \frac{8}{m_{2j+1}^3}.$$

It follows that $\|T\| \geq \frac{\delta}{16} m_{2j+1}^2$, a contradiction for j sufficiently large. \square

Proposition 4.18. *Let $T : X_{ius} \rightarrow X_{ius}$ be a bounded operator. Then there exists $\lambda \in \mathbb{R}$ such that $T - \lambda I$ is strictly singular.*

Proof. By Lemma 4.17 there exists $\lambda \in \mathbb{R}$ and $M \in [\mathbb{N}]$ such that $\lim_{n \in M} \|Te_n - \lambda e_n\| = 0$. Let $\varepsilon > 0$. Passing to a further subsequence $(e_{n_k})_k$, we may assume that $\|Te_{n_k} - \lambda e_{n_k}\| \leq \varepsilon 2^{-k}$ for every $k \in \mathbb{N}$. It follows that the restriction of $T - \lambda I$ to $[e_{n_k}, k \in \mathbb{N}]$ is of norm less than ε . By Proposition 4.16 it follows that $T - \lambda I$ is strictly singular. \square

The following two corollaries are consequences of Proposition 4.18 (see [GM]).

Corollary 4.19. *There does not exist a non trivial projection $P : X_{ius} \rightarrow X_{ius}$.*

Corollary 4.20. *The space X_{ius} is not isomorphic to any proper subspace of it.*

It remains to prove lemmas 4.14 and 4.15. We start with the following.

Lemma 4.21. *Let $j \in \mathbb{N}$, $n_{2j+1} < m_{j_1} < m_{j_2} < \dots < m_{j_{2r}}$ be such that $2r \leq n_{2j+1} < m_{j_1}^{1/2}$. Let also $j_0 \in \mathbb{N}$ be such that $m_{j_0} \neq m_{j_i}$ for every $i = 1, \dots, 2r$ and $m_{j_0}^{1/2} > n_{2j+1}$. Then if $h_1 < \dots < h_{2r} \in K$ are such that $w(h_i) = m_{j_i}$ for every $i = 1, \dots, 2r$, then*

a)

$$(4.22) \quad |(\sum_{k=1}^r \lambda_{2k-1} h_{2k-1} + h_{2k}) (\frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} e_{kl})| < \frac{1}{n_{2j+1}},$$

for every choice of real numbers $(\lambda_{2k-1})_{k=1}^r$ with $|\lambda_{2k-1}| \leq 1$ for every $k \leq r$.

b) If $(x_l)_{l=1}^{n_{j_0}}$ is a $(3, \frac{1}{n_{j_0}})$ -R.I.S of ℓ_1 averages, then

$$(4.23) \quad |(\sum_{k=1}^r \lambda_{2k-1} h_{2k-1} + h_{2k}) (\frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_l)| \leq \frac{1}{n_{2j+1}},$$

for every choice of real numbers $(\lambda_{2k-1})_{k=1}^r$ with $|\lambda_{2k-1}| \leq 1$ for every $k \leq r$.

Proof. We shall give the proof of b) and we shall indicate the minor changes for the proof of a).

From the estimations on the R.I.S, Proposition 4.9, for every $k \leq 2r$ we have that

$$(4.24) \quad |h_k (\frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_l)| \leq \begin{cases} \frac{9}{w(h_k)}, & \text{if } w(h_k) < m_{j_0} \\ \frac{3}{m_r} + \frac{6}{n_{j_0}}, & \text{if } w(h_k) = m_r > m_{j_0}. \end{cases}$$

Using that $m_{j+1} = m_j^5$ for every j and $|\lambda_{2k-1}| \leq 1$ for every $k \leq r$ and (4.24), we get that

$$\begin{aligned} |(\sum_{k=1}^r \lambda_{2k-1} h_{2k-1} + h_{2k}) (\frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_l)| &\leq \sum_{k: w(h_k) < m_{j_0}} \frac{9}{w(h_k)} + \sum_{r > j_0} \frac{3}{m_r} + \frac{12r}{n_{j_0}} \\ &\leq \frac{10}{w(h_1)} + \frac{4}{m_{j_0}^2} + \frac{12r}{n_{j_0}} < \frac{1}{n_{2j+1}}. \end{aligned}$$

For the proof of a) using Lemma 4.2, for the estimations on the basis we get the corresponding inequality to (4.24), from which follows inequality (4.22). \square

Proof of Lemma 4.14. Let $\chi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$ be a depended sequence and $\phi = (y_1, f_1, y_2, f_2, \dots, y_{n_{2j+1}}, f_{n_{2j+1}})$ the special sequence associated to χ . In the rest of the proof we shall assume that $\chi = \phi$. The general proof follows by slight and obvious modifications of the present proof. Hence we assume that $\phi = (x_1, f_1, \dots, x_{n_{2j+1}}, f_{n_{2j+1}})$.

From Lemma 4.2 and Remark 4.13a) it follows that the sequence $(m_{j_i} x_i)_{i=1}^{n_{2j+1}}$ satisfies assumptions a), c) of the basic inequality for $C = 2$. Furthermore the properties of the function σ yield that assumption b) is also satisfied for $\varepsilon = 1/n_{2j+1}$.

The rest of the proof is devoted to establish that the sequence $(m_{j_i} x_i)_i$ satisfies the crucial condition d) for $m_{j_0} = m_{2j+1}$ and $(b_i)_i = (\frac{(-1)^{i+1}}{n_{2j+1}})_i$.

First we consider $f \in K_\phi$. Then f is of the form

$$f = E(\frac{\varepsilon}{m_{2j+1}} (\lambda_{f'_2} f_1 + f'_2 + \dots + \lambda_{f'_{n_{2j+1}}} f_{n_{2j+1}-1} + f'_{n_{2j+1}})),$$

where $\varepsilon \in \{-1, 1\}$ and E an interval of \mathbb{N} . Let us recall that $w(f'_{2i}) = w(f_{2i})$ and $\text{supp}(f'_{2i}) = \text{supp}(f_{2i})$ and therefore $\text{range}(f'_{2i}) \cap \text{range}(x_k) = \emptyset$ for every $k \neq 2i$. Let

$$i_0 = \min\{i \leq n_{2j+1}/2 : \text{supp}(f) \cap (\text{range}(x_{2i-1}) \cup \text{range}(x_{2i})) \neq \emptyset\}.$$

Then

$$(4.25) \quad \left| f\left(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i\right) \right| = \left| E \frac{1}{m_{2j+1}} \sum_{k=1}^{n_{2j+1}/2} (\lambda_{f'_{2k}} f_{2k-1} + f'_{2k}) \left(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i\right) \right| \leq$$

$$\frac{1}{m_{2j+1}} \left| \lambda_{f'_{2i_0}} E f_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E f'_{2i_0}(m_{j_{2i_0}} x_{2i_0}) \right|$$

$$(4.26) \quad + \frac{1}{m_{2j+1}} \left| \sum_{i=i_0+1}^{n_{2j+1}/2} (\lambda_{f'_{2i}} f_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - f'_{2i}(m_{j_{2i}} x_{2i})) \right|.$$

To estimate the sum in (4.25) and (4.26), we partition the set $\{i_0, \dots, n_{2j+1}/2\}$ into two sets A and B , where $A = \{i : f'_{2i}(x_{2i}) \neq 0\}$ and B is its complement. For every $i \in A$, $i > i_0$, using that $\lambda_{f'_{2i}} = f'_{2i}(m_{j_{2i}} x_{2i})$, we have that

$$(4.27) \quad \lambda_{f'_{2i}} f_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - f'_{2i}(m_{j_{2i}} x_{2i}) = f'_{2i}(m_{j_{2i}} x_{2i}) - f'_{2i}(m_{j_{2i}} x_{2i}) = 0.$$

For every $i \in B$ we have that $f'_{2i}(x_{2i}) = 0$, and therefore, $|\lambda_{f'_{2i}}| = \frac{1}{n_{2j+1}^2}$, see (2.6). It follows that, for every $i \in B$, $i > i_0$

$$(4.28) \quad |\lambda_{f'_{2i}} f_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - f'_{2i}(m_{j_{2i}} x_{2i})| = |\lambda_{f'_{2i}}| = \frac{1}{n_{2j+1}^2}.$$

For the sum $|\lambda_{f'_{2i_0}} E f_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E f'_{2i_0}(m_{j_{2i_0}} x_{2i_0})|$ distinguishing whether or not $E f_{2i_0-1} = 0$ and whether $i_0 \in A$ or $i_0 \in B$, it follows easily using the previous arguments that

$$(4.29) \quad |\lambda_{f'_{2i_0}} E f_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E f'_{2i_0}(m_{j_{2i_0}} x_{2i_0})| \leq 1$$

Summing up (4.27)-(4.29) we have that

$$(4.30) \quad \left| f\left(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i\right) \right| \leq \frac{1}{m_{2j+1}} \left(\frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}^2} \right) < \frac{1}{n_{2j+1}}.$$

Consider now a *special sequence* $\psi = (y_1, g_1, y_2, g_2, \dots, y_{n_{2j+1}}, g_{n_{2j+1}})$. Let $i_1 = \min\{i \in \{1, \dots, n_{2j+1}\} : y_i \neq x_i \text{ or } g_i \neq f_i\}$, and $k_0 \in \mathbb{N}$ such that $i_1 = 2k_0 - 1$ or $2k_0$.

Consider a functional $g \in K_\psi$ which is defined from this special sequence. Then we have that

$$g = E\left(\frac{1}{m_{2j+1}} (\lambda_{g'_2} g_1 + g'_2 + \dots + \lambda_{g'_{n_{2j+1}}} g_{n_{2j+1}-1} + g'_{n_{2j+1}})\right),$$

where E is an interval of \mathbb{N} and $w(g'_{2i}) = w(g_{2i})$ for every $i \leq n_{2j+1}/2$. Observe that $\text{range}(x_i) \cap \text{range}(g_k) = \emptyset$ for every $i \geq i_1$ and every $k < i_1$. Let

$$i_0 = \min\{i \leq n_{2j+1}/2 : \text{supp}(g) \cap (\text{range}(x_{2i-1}) \cup \text{range}(x_{2i})) \neq \emptyset\}.$$

Let $i_0 < k_0$. Then

$$\begin{aligned}
(4.31) \quad & |g(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i)| \leq \\
(4.32) \quad & \frac{1}{m_{2j+1}} \left(|E \lambda_{g'_{2i_0}} g_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E g'_{2i_0}(m_{j_{2i_0}} x_{2i_0})| \right. \\
& \quad \left. + \left| \sum_{i=i_0+1}^{k_0-1} (\lambda_{g'_{2i}} g_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - g'_{2i}(m_{j_{2i}} x_{2i})) \right| \right) \\
(4.33) \quad & + \frac{1}{m_{2j+1}} \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k}) \left(\sum_{i \geq i_0} m_{j_{2i-1}} x_{2i-1} - m_{j_{2i}} x_{2i} \right) \right|.
\end{aligned}$$

where the sum in (4.32) makes sense when $i_0 < k_0 - 1$. If $i_0 \geq k_0$ we get that

$$|g(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i)| \leq \frac{1}{m_{2j+1}} |E \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k}) \left(\sum_{i \geq i_0} m_{j_{2i-1}} x_{2i-1} - m_{j_{2i}} x_{2i} \right)|.$$

The proof of the upper estimation for the two cases is almost identical, so we shall give the proof in the case $i_0 < k_0$.

As in the previous case, for the sum in (4.31),(4.32) we have that

$$\begin{aligned}
(4.34) \quad & |E \lambda_{g'_{2i_0}} g_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E g'_{2i_0}(m_{j_{2i_0}} x_{2i_0})| + \\
& \left| \sum_{i=i_0+1}^{k_0-1} (\lambda_{g'_{2i}} g_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - g'_{2i}(m_{j_{2i}} x_{2i})) \right| \leq 2.
\end{aligned}$$

To estimate the sum in (4.33), first we observe that from the injectivity of σ it follows that there exists at most one $k \geq i_1$ such that

$$w(g_k) \in \{m_{j_i} : i_1 \leq i \leq n_{2j+1}\}.$$

Let $2i-1 \geq i_1$ be such that $m_{j_{2i-1}} \neq w(g_k)$ for every $k \geq i_1$. Then functionals g_{2k-1}, g'_{2k} , $k \geq k_0$ satisfy the assumptions of Lemma 4.21, and therefore we get that

$$(4.35) \quad \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k})(m_{j_{2i-1}} x_{2i-1}) \right| \leq \frac{1}{n_{2j+1}}.$$

Also for every $2i \geq i_1$ such that $m_{j_{2i}} \neq w(g_k)$ for every $k \geq i_1$, the functionals g_{2k-1}, g'_{2k} , $k \geq k_0$ satisfy the assumptions of Lemma 4.21, and therefore we get that

$$(4.36) \quad \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k})(m_{j_{2i}} x_{2i}) \right| \leq \frac{1}{n_{2j+1}}.$$

For the unique $i \geq i_1$, such that there exists $k \geq i_1$ and $w(g_k) = m_{j_i}$, if such an i exists, we have that, using Lemma 4.21

$$(4.37) \quad \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k})(m_{j_i} x_i) \right| \leq 1 + \frac{1}{n_{2j+1}}.$$

Now we distinguish if $i_1 = 2k_0 - 1$ or $i_1 = 2k_0$. If $i_1 = 2k_0 - 1$, we have that $\text{range}(g_k) \cap \text{range}(x_i) = \emptyset$ for every $k < 2k_0 - 1$ and every $i \geq 2k_0 - 1$, and from (4.35)-(4.37) we get

that

$$(4.38) \quad \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k}) \left(\frac{1}{n_{2j+1}} \sum_{i=2k_0-1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i \right) \right| \leq \frac{1}{n_{2j+1}} \left(1 + \frac{1}{n_{2j+1}} + \frac{n_{2j+1}}{n_{2j+1}} \right) < \frac{3}{n_{2j+1}}.$$

If $i_1 = 2k_0$ then we have that $\text{range}(x_{2k_0-1}) \cap \text{range}(g_k) = \emptyset$ for every $k \geq 2k_0$ and $k < 2k_0 - 1$, and from (4.35)-(4.37) we get that

$$(4.39) \quad \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k}) \left(\frac{1}{n_{2j+1}} \sum_{i=2k_0-1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i \right) \right| \leq \frac{1}{n_{2j+1}} \left(|\lambda_{g'_{2k_0-1}} g_{2k_0-1} (m_{j_{2k_0-1}} x_{2k_0-1})| \right. \\ \left. + \left| \sum_{k \geq k_0} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k}) \left(\sum_{i=2k_0}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i \right) \right| \right) \leq \frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \left(1 + \frac{1}{n_{2j+1}} + \frac{n_{2j+1}}{n_{2j+1}} \right) < \frac{4}{n_{2j+1}}.$$

From (4.34), (4.38) and (4.39) we get that

$$(4.40) \quad \left| g \left(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_{j_i} x_i \right) \right| \leq \frac{1}{m_{2j+1}} \left(\frac{2}{n_{2j+1}} + \frac{4}{n_{2j+1}} \right) < \frac{1}{n_{2j+1}}.$$

The inequalities (4.30) and (4.40) yield that indeed condition d) is satisfied for $\varepsilon = 1/n_{2j+1}$. Proposition 4.9 (2) derives the desired result and the proof is complete. \square

Proof of Lemma 4.15. To prove this we shall follow similar arguments as in the proof of Lemma 4.14. We shall establish conditions a), b), c) and d) of the basic inequality, for $C = 2$, $\varepsilon = \frac{1}{n_{2j+1}}$ and $m_{j_0} = m_{2j+1}$. Lemma 4.2 yields that the sequence $(y_{2i})_i$ satisfies the assumptions a) and c) of the basic inequality for $C = 2$. Furthermore the properties of the function σ yield that assumption b) is also satisfied for $\varepsilon = 1/n_{2j+1}$.

To establish condition d) we shall show that for every $f \in K$ with $w(f) = m_{2j+1}$, it holds that

$$\left| f \left(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right) \right| \leq \frac{1}{m_{2j+1}} \left(\frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \right) < \frac{1}{n_{2j+1}}.$$

First let us observe that for every $f \in K_\phi$, $f = E \frac{1}{m_{2j+1}} \sum_{k=1}^{\frac{n_{2j+1}}{2}} (\lambda_{f'_{2k}} f_{2k-1} + f'_{2k})$ it holds

that $f \left(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right) = 0$. This is due to $\text{supp} f'_{2i} = \text{supp} f_{2i}$ and $\text{supp}(f_{2i-1}) < y_{2i} < \text{supp}(f_{2i+1})$ for every $i \leq n_{2j+1}/2$.

Let $\phi = (z_1, g_1, z_2, g_2, \dots, z_{n_{2j+1}}, g_{n_{2j+1}})$ be a *special sequence* of length n_{2j+1} and let $f = E \frac{1}{m_{2j+1}} \sum_{k=1}^{\frac{n_{2j+1}}{2}} (\lambda_{g'_{2k}} g_{2k-1} + g'_{2k})$ belonging to K_ϕ . Without loss of generality we may assume that $E = \mathbb{N}$. Let $i_1 = \min\{i \leq n_{2j+1} : z_i \neq x_i \text{ or } f_i \neq g_i\}$, and $k_0 \in \mathbb{N}$ such that $i_1 = 2k_0 - 1$ or $i_1 = 2k_0$. Observe that $\text{range}(g_k) \cap \text{range}(y_{2i}) = \emptyset$ for every $k < i_1$ and every $2i \geq i_1$.

From the injectivity of σ , it follows that there exists at most one $k \geq i_1$ such that

$$w(g_k) \in \{m_{j_i} : i_1 \leq i \leq n_{2j+1}\}.$$

Let $2i \geq i_1$ such that $w(g_k) \neq m_{j_{2i}}$ for all $k \geq i_1$. Then the functionals $g_{2k-1}, g'_{2k}, k \geq k_0$ satisfy the assumptions of Lemma 4.21(a), and therefore it follows that

$$(4.41) \quad |(\sum_{k \geq k_0} \lambda_{g'_{2k}} g_{2k-1} + g'_{2k})(y_{2i})| < \frac{1}{n_{2j+1}}.$$

For the unique $2i \geq i_1$ such that there exists $k \geq i_1$ with $w(g_k) = m_{j_{2i}}$, if such $2i$ exists, we have that

$$(4.42) \quad |(\sum_{k \geq k_0} \lambda_{g'_{2k}} g_{2k-1} + g'_{2k})(y_{2i})| < 1 + \frac{1}{n_{2j+1}}.$$

Summing up (4.41)-(4.42) we get that

$$(4.43) \quad |f(\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i})| \leq \frac{1}{m_{2j+1}} (\frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}}) < \frac{1}{n_{2j+1}}.$$

Inequality (4.43) implies that condition d) of the basic inequality is fulfilled, and Proposition 4.9 yields the desired result. \square

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